Phase-frequency synchronization in a chain of periodic oscillators in the presence of noise and harmonic forcings

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(Received 1 September 2000; published 27 February 2001)

We study numerically the effects of noise and periodic forcings on cluster synchronization in a chain of Van der Pol oscillators. We generalize the notion of effective synchronization to the case of a spatially extended system. It is shown that the structure of synchronized clusters can be effectively controlled by applying local external forcings. The effect of amplitude relations on the phase dynamics is also explored.

DOI: 10.1103/PhysRevE.63.036225 PACS number(s): 05.45.Xt

I. INTRODUCTION

The phenomenon of synchronization plays an important role in the behavior of ensembles of interacting nonlinear oscillators. This effect provides the basis for self-organization of ensemble dynamics and is associated with a variety of phenomena, such as multistability, growth restriction of the Kolmogorov entropy and attractor dimension, spatiotemporal structure formation, etc. The theory of synchronization, originally proposed for quasiperiodic oscillations [1–4], was generalized to a wide range of systems including chaotic [5–10] and stochastic [11,12] ones.

Phase synchronization in ensembles of locally and globally coupled interacting periodic oscillators has been studied for a long time but these investigations still attract the growing interest of many researchers [13,4,14–24]. Ensembles of periodic oscillators have found wide applications in mathematical modeling of physical [22,25–29], chemical [13,4], and biological [30–34] processes.

It is known that fluctuations are inevitably present in real ensembles and a parameter mismatch (random or definitely specified) of partial systems also takes place. Effects of noise and frequency mismatch on phase locking in an ensemble of oscillators are considered in [14,15,17–21,24]. The presence of a linear gradient of native unperturbed frequencies along the medium consisting of locally coupled oscillators leads to the formation of so-called clusters of phase synchronization [17,24].

Recently, numerous works have appeared devoted to the study of ensembles of chaotic oscillators [25,35–43]. It has been shown that the synchronization effect also plays an important role in the dynamics of chaotic ensembles. They demonstrate a number of phenomena that appear to be quite similar to those that have occurred in ensembles of periodic oscillators. Particularly, effects of phase locking and cluster-phase synchronization have been found in ensembles of chaotic oscillators [42,43]. This fact testifies that the effect of synchronization is generic for a variety of oscillatory systems.

However, in this research direction there remain a number of unresolved problems that require special attention. How stable can cluster synchronization be under the influence of fluctuations? Is it possible to generalize the notion of effective synchronization of self-sustained oscillations in the presence of noise [44,45] to spatially extended systems? What effect can a local external harmonic signal have on the clusters of synchronization and is it possible to control the clusters' parameters by means of external forcing? It is interesting to elucidate how significant it may be if a variation of instantaneous amplitude values of oscillators is taken into consideration? Can the behavior of an ensemble be qualitatively described by the phase equations only? We try to answer the above stated questions and this is the main objective of this paper.

The paper is organized as follows. In Sec. II we describe the model and the problems that we address in the paper. In Sec. III we study the effect of noise on phase synchronization in a chain of nonidentical Van der Pol oscillators. The possibility of controlling cluster synchronization by means of local external forcing is discussed in Sec. IV. In Sec. V we explore the peculiarities of behavior of the chain of Van der Pol oscillators described by the phase equations only. And finally, we give our conclusions in Sec. VI.

II. MODEL AND PROBLEM STATEMENT

The model under study is a chain of Van der Pol oscillators, being similar to that considered in [24] and including, in the general case, additive noise and harmonic forcings on the chain elements. The chain is described by a system of equations which, in a truncated form, are as follows:

$$\begin{aligned}\dot{\rho}_j &= r(1-\rho_j^2)\rho_j + g[\rho_{j-1}\cos(\phi_j-\phi_{j-1}) \\ &\quad + \rho_{j+1}\cos(\phi_{j+1}-\phi_j) - 2\rho_j] + F_j(t), \\ \dot{\phi}_j &= \omega_j + g\left[\frac{\rho_{j+1}}{\rho_j}\sin(\phi_{j+1}-\phi_j) - \frac{\rho_{j-1}}{\rho_j}\sin(\phi_j-\phi_{j-1})\right] \\ &\quad + P_j(t), \qquad j=1,2,3,\ldots,m,\end{aligned} \tag{1}$$

where j is the number of an oscillator, representing a discrete spatial coordinate, ρ_j and ϕ_j are the amplitude and the phase of oscillations of the jth oscillator, respectively. $F_j(t)$ and $P_j(t)$ denote the forcings acting on the jth chain element. Each of them can be presented as the sum of harmonic and noisy terms,

$$F_j(t)=A_j \sin(\omega_{\rm ex}^j t-\phi_j)+\sqrt{2D}\xi_j(t),$$

$$P_j(t)=-\frac{A_j}{\rho_j}\cos(\omega_{\rm ex}^j t-\phi_j)+\frac{\sqrt{2D}}{\rho_j}\eta_j(t), \quad j=1,2,\ldots,m.$$

The boundary conditions were chosen to correspond to a free-ended chain, i.e., $\rho_0=\rho_1$, $\phi_0=\phi_1$, $\rho_{m+1}=\rho_m$, $\phi_{m+1}=\phi_m$.

The model (1) has the following parameters: r is the excitation parameter (in computations, we fix $r=0.5$), ω_j is the unperturbed frequency of the jth oscillator, i.e., oscillation frequency without coupling and external forcings, g is the coupling parameter, A_j and $\omega_{\rm ex}^j$ denote the normalized amplitude and the frequency of the harmonic force acting on the jth element, and $\xi_j(t)$ and $\eta_j(t)$ are assumed to be identical, uncorrelated Gaussian white noise sources with zero means and with the same intensity D.[1] In our work we are dealing with a case of linear dependencies of the unperturbed frequencies on the spatial coordinate j, i.e., $\omega_j=\omega_1+(j-1)\Delta$, where Δ is the frequency mismatch of two neighboring oscillators. The peculiarities of the chain dynamics do not depend on the choice of the frequency origin. Therefore in our computations, we set $\omega_1=0$, thus shifting the origin of frequencies by the value of the unperturbed frequency of the first oscillator.

We study numerically the chain (1) with $m=100$ elements using a fourth-order Runge-Kutta routine. In the course of numerical experiments, we analyze the dynamics of each element, estimate the variation of phases ϕ_j during a large enough time T, and compute the average (perturbed) frequencies $\tilde{\omega}_j$ of the partial oscillators,

$$\tilde{\omega}_j=\langle \dot{\phi}_j(t)\rangle=\lim_{T\to\infty}\frac{\phi_j(t_0+T)-\phi_j(t_0)}{T}. \qquad (2)$$

The angle brackets mean time averaging. The initial conditions for the oscillators are chosen to be close to homogeneous ones with a small random dispersion within $\delta=0.1$.

In this paper we mainly address the issue concerning the effect of noise and external forcing on cluster synchronization in a chain of nonidentical Van der Pol oscillators. We explore (i) the pure noise effect on all chain elements, (ii) the same situation but with a local synchronous harmonic forcing on a chosen element, and (iii) the effect of local harmonic forcing on chosen elements of the chain with no noise added.

[1] In fact, in numerical experiments the same pseudorandom number generator was used having a Gaussian distribution. Successive values produced by the generator may be treated as practically independent . To make sure that the noise disturbances are uncorrelated, the noise source added to each subsequent element of the chain was shifted with respect to the previous one by five iterations of the pseudorandom number generator.

III. EFFECT OF NOISE ON PHASE SYNCHRONIZATION IN A CHAIN OF NONIDENTICAL VAN der POL OSCILLATORS

We consider an autonomous chain ($A_j=0$ for any j) of nonidentical Van der Pol oscillators with a linear frequency gradient along the spatial coordinate j. For such a chain arrangement, one can observe cluster phase synchronization in a certain range of coupling parameter g values [24]. Partial oscillators exhibit quasiperiodic oscillations $x_j(t)=\rho_j(t)\cos\phi_j(t)$ and $y_j(t)=\rho_j(t)\sin\phi_j(t)$, and the number of frequencies in the spectrum of oscillations is determined by the number of synchronized clusters. Figures 1(a) and 1(b) illustrates (x_j, y_j) projections of oscillations in the regime of cluster synchronization that are characteristic for the center and the boundary of a cluster, respectively. If we consider oscillators within the same cluster, then a representative point rotates about the origin $x_j=0$, $y_j=0$, on an average, with the same frequency and a bounded phase shift. Oscillators belonging to different clusters have distinct rotation frequencies. Consequently, the form of phase projections (x_j, x_k) is qualitatively different when the jth and kth oscillators belong to one cluster [Fig. 1(c)] and to different clusters [Fig. 1(d)].

Now we are going to elucidate how the noise influences the cluster synchronization. We fix $\Delta=0.002$ and compute the distribution of perturbed frequencies $\tilde{\omega}_j$ of oscillators along the chain without and with noise. The calculation results, shown in part I of Figs. 2(a) and 2(b) for two different values of the coupling parameter g, clearly demonstrate the effect of cluster synchronization in the noise-free chain ($D=0$) and completely correspond to the analogous results presented in [24].

Now consider the case when all oscillators are subjected to noisy perturbations. Parts (II) and (III) of Figs. 2(d) and 2(e) present the distributions of the perturbed frequencies for two different noise intensities $D=0.000\,01$ and $D=0.001$, respectively. It is clearly seen that for both coupling parameters, the clusters of synchronization are destroyed as the noise intensity increases. If the noise is weak, the clusters' boundaries are only smoothening slightly [graphs (II)]. Both smoothening and gradual destruction of the clusters begin with the chain center. For sufficiently large noise [graphs (III)], all middle clusters are completely destroyed. However, our computations have shown that the first and the last clusters appear to be highly stable to noisy disturbances and only a very strong noise is needed to destroy them.

It is interesting to explore how the addition of local synchronous forcing can affect noise-induced cluster destruction. For this purpose, the 50th element of the chain is subjected to external harmonic forcing with amplitude $A_{50}=1$ and frequency $\omega_{\rm ex}^{50}=(j-1)\Delta$, $j=50$, being equal to the unperturbed frequency of the chosen oscillator. The other oscillators remain unforced, i.e., $A_j=0$ for $j\neq 50$. We apply noise of a sufficiently large intensity $D=0.001$ so that all synchronized clusters are destroyed, except the first and the last ones. The results of synchronous action are shown in part (IV) of Figs. 2(a) and 2(b). It can be seen that the

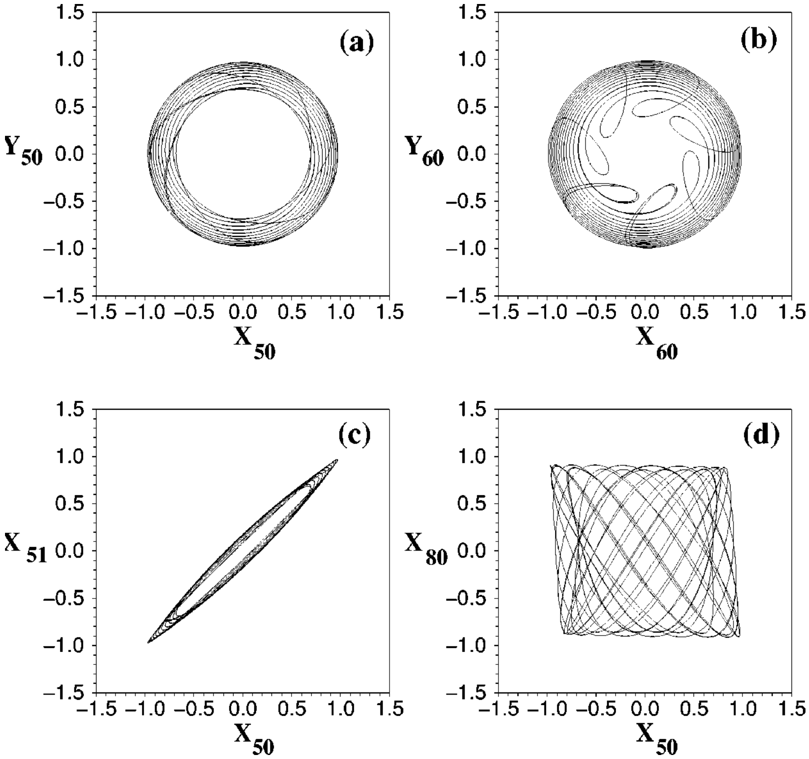

FIG. 1. Phase projections of oscillations of partial oscillators in the regime of cluster synchronization for $\Delta = 0.002$ and $g = 3.8$.

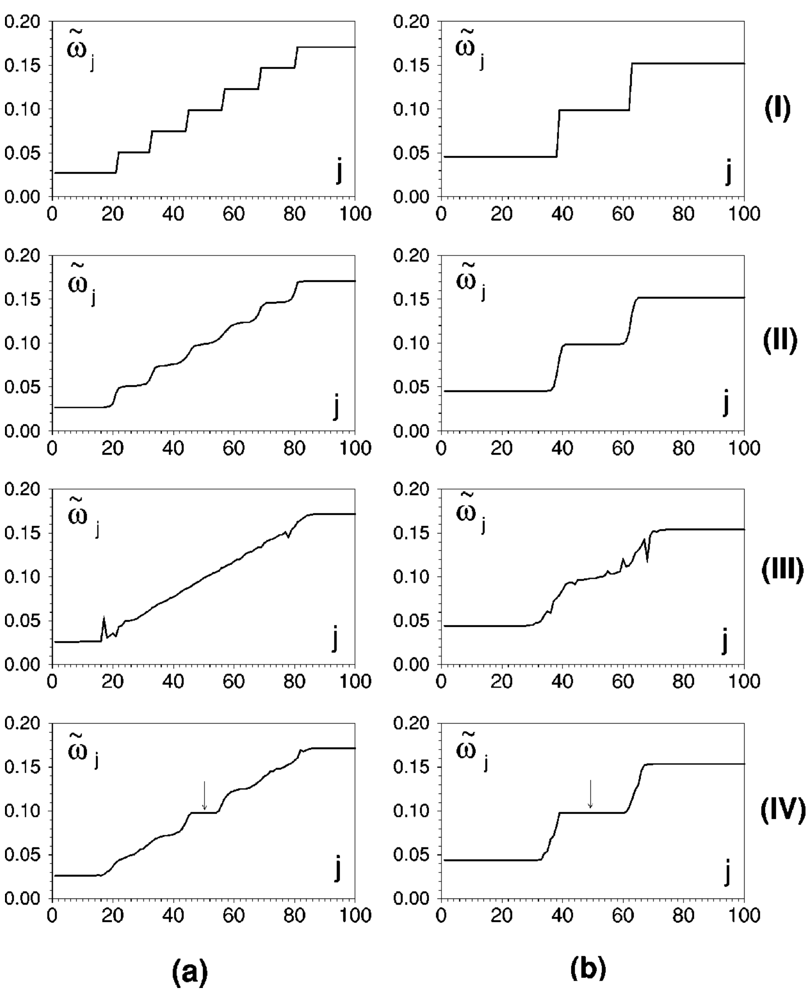

FIG. 2. Distributions of the perturbed oscillator frequencies for $\Delta = 0.002$ and for different strengths of coupling: (a) $g = 0.55$; (b) $g = 3.8$. Dependencies (I), (II) and (III) are obtained for the autonomous chain in the presence of noise with intensity $D = 0$, $D = 0.000\,01$, and $D = 0.001$, respectively. Plots (IV) reflect the results of action of synchronous forcing applied to the 50th element, as indicated by arrows, in the presence of noise with intensity $D = 0.001$.

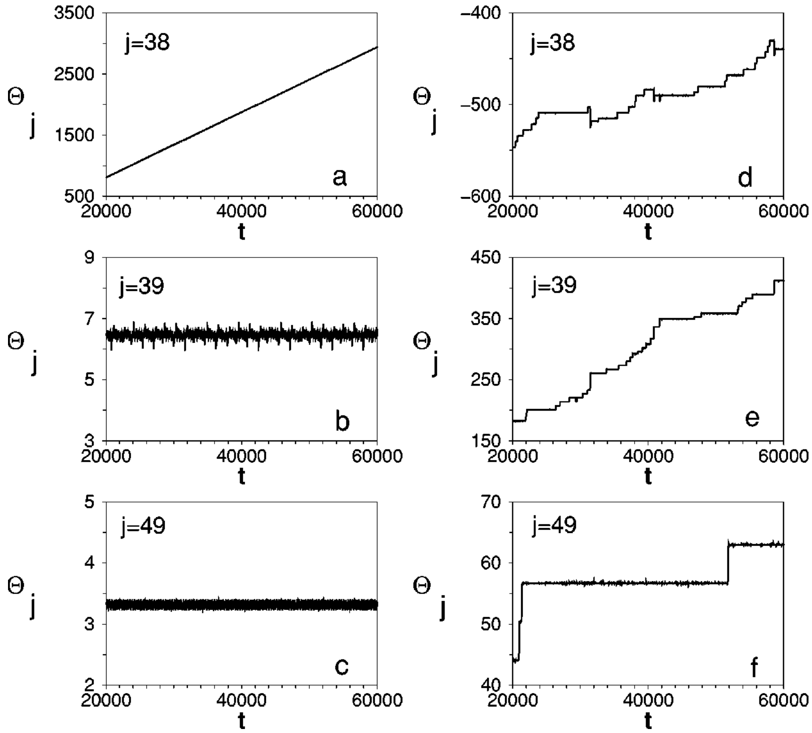

FIG. 3. Temporal dependencies of phase difference $\theta_j(t)$ of neighboring oscillators without noise near the cluster's boundary (a), (b), in the center of the cluster (c), and in the presence of noise of intensity $D=0.001$ (d), (e), and (f). The parameters are $\Delta=0.002$ and $g=3.8$.

external forcing added even to a single element of the chain enables to partially recover the structure of the noise-destroyed clusters.

The effect of noise on cluster synchronization can be more clearly understood by considering how the phase differences $\theta_j(t)=\phi_{j+1}(t)-\phi_j(t)$ of neighboring oscillators, located near the cluster's boundary, change with time without and in the presence of noise. The corresponding dependencies are shown in Fig. 3 and demonstrate how the sharply-defined boundary of phase synchronization can be destroyed in the presence of fluctuations. Without noise, the phase difference of oscillators belonging to different clusters increases, on an average, linearly as the time goes on [Fig. 3(a)]. At the same time, the phase difference remains unchanged if the oscillators considered belong to the same cluster (we exclude oscillations within the interval $[-\pi,\pi]$ with respect to the mean value) [see Figs. 3(b) and 3(c)]. When the noise is added, the phase difference of any of the neighboring oscillators grows indefinitely with time but this growth is not linear for any value of j [Fig. 3(d)–3(f)]. The average growth rate of the phase difference is different for different j. This fact allows one to find certain segments of the chain for which this rate is low. Hence, we can identify clusters of effective phase synchronization in the presence of noise [45].

The clusters' boundaries in the presence of noise can be estimated by using the effective diffusion coefficient D_{eff} of the phase difference of neighboring oscillators [44]. D_{eff} defines the average rate with which the variance $\sigma^2_{\theta_j}(t)$ of phase difference θ_j increases in time. Its mean value can be calculated as follows:

$$D_{\mathrm{eff}}(j)=\lim_{t\to\infty}\frac{1}{2}\frac{\sigma^2_{\theta_j}(t)}{t},\quad \sigma^2_{\theta_j}(t)=\langle\theta_j^2(t)\rangle-\langle\theta_j(t)\rangle^2. \tag{3}$$

We compute the effective diffusion coefficient versus the spatial coordinate within one cluster ($39\leqslant j\leqslant 62$) for three different values of noise intensity D. Numerical results are presented in Fig. 4. They testify to a gradual destruction of the cluster's boundaries as the noise intensity increases. One can note that the dependence $D_{\mathrm{eff}}(j)$ is quite similar (taking into account that j is a discrete variable) to a well-known dependence, of the diffusion coefficient of the phase difference between a self-sustained system and external forcing versus detuning. The cluster's boundaries of effective phase synchronization can be defined by specifying some tolerable level of the diffusion coefficient $D_{\mathrm{eff}}^{\max}$. In this case oscillators for which $D_{\mathrm{eff}}\leqslant D_{\mathrm{eff}}^{\max}$ can be considered as belonging to the same cluster. Such a determination of the cluster's boundaries is arbitrary enough since the value of $D_{\mathrm{eff}}^{\max}$ can be given in different ways depending on a particular task. How-

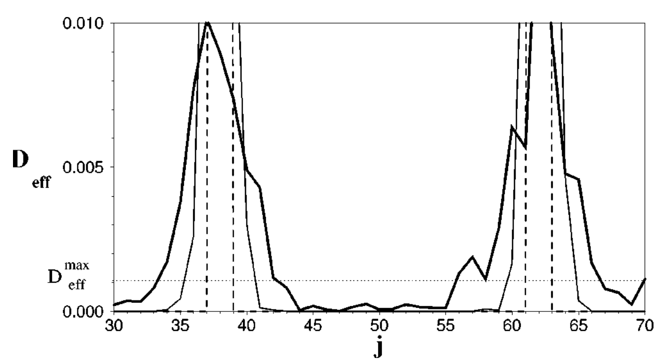

FIG. 4. Effective diffusion coefficient D_{eff} as a function of spatial coordinate j for $D=0$ (thin dashed line), $D=0.000\,01$ (thin solid line), and $D=0.0001$ (thick solid line). The horizontal dotted line marks the level of $D_{\mathrm{eff}}^{\max}$, defining the cluster's boundaries. The detuning and the coupling strength are $\Delta=0.002$ and $g=3.8$.

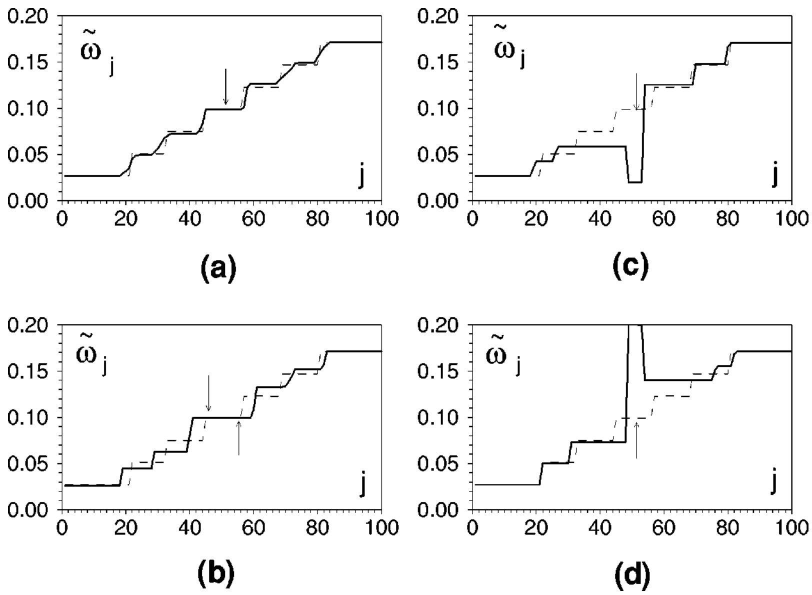

FIG. 5. Structure of synchronized clusters in the chain of Van der Pol oscillators for $\Delta = 0.002$ and $g = 0.55$ in the presence of (a) synchronous forcing on the central element of the middle cluster, (b) synchronous forcing on the boundary elements of the middle cluster, and asynchronous forcing with smaller (c) and larger (d) frequencies acting on the central element of the middle cluster. Forcing amplitudes are taken to be equal to unity. Arrows indicate the places where external forcings are applied. For comparison, the original structure of synchronized clusters is shown by a dashed line.

ever, in any case the length of a cluster decreases with increasing noise intensity. For example, given $D_{\mathrm{eff}}^{\max} = 0.001$, the boundaries of the cluster shown in Fig. 4 for $D = 0.0001$ correspond to the 44th and the 56th oscillators.

IV. CONTROLLING CLUSTER SYNCHRONIZATION BY MEANS OF LOCAL EXTERNAL FORCING

Now we are going to study how local external forcings may influence the dynamics of a chain with no noise added. This problem seems to be quite interesting since its solution can provide us the possibility to control cluster synchronization. We consider a noise-free chain of Van der Pol oscillators with a linear frequency gradient along the chain. Our aim is to study the influence of external harmonic forcings with different frequencies ω_{ex}^{j} on the structure of clusters in this chain. In our computation, we fix $\Delta = 0.002$ and $g = 0.55$. Figure 5(a) illustrates how the structure of clusters changes when the local forcing with amplitude $A = 1$ is applied to the central element of the middle cluster ($j = 51$). For comparison, the structure of clusters without forcing is shown in Fig. 5 by a dashed line. As can be seen, the distribution of perturbed partial frequencies remains practically the same under synchronous forcing and only the boundaries between the clusters become less marked. Another picture is observed when the external forcing is applied to the boundary elements ($j = 44$ and $j = 54$) of the middle cluster [Fig. 5(b)]. First, the lengths of all seven synchronization clusters change, especially that of the forced cluster, which is considerably increased from 11 to 18 elements ($41 \leqslant j \leqslant 59$). Secondly, the partial frequencies $\tilde{\omega}_j$ change for 2,3 and 5,6 clusters.

Now we take an asynchronous forcing with amplitude $A_j = 1$ and apply it to the central element ($j = 51$) of the same middle cluster. The results of the forcing action are shown in Figs. 5(c) and 5(d) for two different forcing frequencies $\omega_{\mathrm{ex}}^{51} = 0.02$ and $\omega_{\mathrm{ex}}^{51} = 0.2$, respectively. These values of ω_{ex}^{j} were chosen to be accordingly much less and much larger than the characteristic frequency of the given cluster. The graphs of Figs. 5(c) and 5(d) demonstrate effects being qualitatively similar to those described for the synchronous forcing. The number of clusters remains the same but their length and partial frequencies $\tilde{\omega}_j$ change. However, there are noticeable differences as compared to the synchronous case. First, there occurs external synchronization of the forced and a few neighboring oscillators at the controlling signal frequency. Secondly, the length of one of the neighboring clusters increases abruptly. When $\omega_{\mathrm{ex}}^{51} \ll \tilde{\omega}_{51}$, this is the left cluster from the central one [Fig. 5(c)] and when $\omega_{\mathrm{ex}}^{51} \gg \tilde{\omega}_{51}$, the right cluster from the central one becomes larger [Fig. 5(d)].

The numeric results, shown in Fig. 5, reflect new nonlinear effects in a chain of coupled oscillators under external forcing. Unfortunately, we are as yet unable to fully explain these results from the physical point of view.

Clusters of phase synchronization can also be observed in a chain of identical elements ($\Delta = 0$) when chosen elements are subjected to external signals with frequencies linearly depending on the spatial coordinate j. Examples of such forced cluster synchronization are presented in Figs. 6(a) and 6(b) for two strengths of coupling $g = 1$ and $g = 2$, respectively; the parameters of external forcings are $A_j = 2$ and $\omega_{\mathrm{ex}}^{j} = (j-1)\Delta$, where $\Delta = 0.001$. The external signals are injected in the elements with $j = 10k$, $k = 1,2, \ldots ,10$ in case (a) and in the elements with $j = 10 + 15k$, $k = 1,2, \ldots ,6$ in case (b).

V. PHASE DYNAMICS APPROACH

In the previous sections we have numerically studied the chain of Van der Pol oscillators, which is described by the system of truncated equations (1) where amplitude and phase dynamics are combined. However in many cases only phase equations are often used assuming amplitudes to be equal and constant in time. Such an approach allows one to qualitatively describe effects of frequency and phase locking and to simplify numerical simulation. Besides, in some cases the

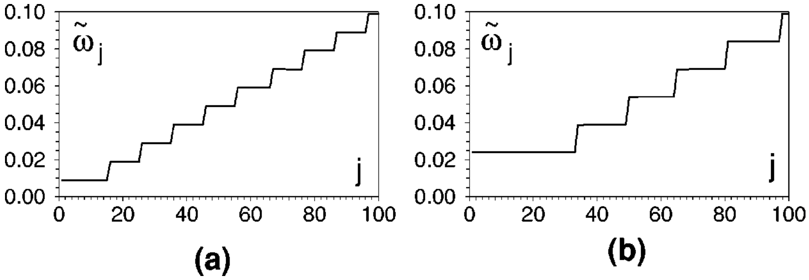

FIG. 6. Examples of forced cluster synchronization in the chain of identical Van der Pol oscillators. External signals with amplitudes $A_j=2$ and frequencies $\omega_{\text{ex}}^j=(j-1)\Delta$, where $\Delta=0.001$, are applied to the chain elements with (a) $j=10k$, $k=1,2,\ldots,10$ for $g=1$, and (b) $j=10+15k$, $k=1,2,\ldots,6$ for $g=2$.

problem can be solved analytically using the phase equations only [13,4,17,20,21,23,46]. Nevertheless, the dynamics of an ensemble may be distorted and some effects may be lost such as, for example, ''oscillator death'' [19,18,24,47] if the amplitude dynamics is excluded from consideration. In particular, as emphasized in [24], amplitude effects may influence the cluster structure formation. To reveal such an effect, we analyze first-cluster synchronization in the unforced chain described by the phase equations only and then compare it with relevant results obtained for the full system of truncated equations (1). The system of phase equations can easily be derived from Eq. (1) by setting $\rho_j=1$ for any j. This means that the amplitudes of all oscillators are taken to be equal to their unperturbed value. The system of phase equations reads

$$\dot{\phi}_j=\omega_1+(j-1)\Delta+g[\sin(\phi_{j+1}-\phi_j)-\sin(\phi_j-\phi_{j-1})] + \sqrt{2D}\,\eta_j(t), \quad j=1,2,\ldots,m. \tag{4}$$

The boundary conditions corresponding to the free ends are $\phi_0=\phi_1$, $\phi_{m+1}=\phi_m$. The detuning is fixed as $\Delta=0.002$. The frequency distributions calculated from Eq. (4) are shown in Fig. 7 for different strengths of coupling. The first three plots correspond to the noise-free case. In Fig. 7(a) illustrating the frequency distribution for $g=0.55$, only two clusters can be observed being formed at the boundaries of the chain. The analogous distribution, presented in part (I) of Fig. 2(a) for the full system [Eq. (1)], reflects a more rich synchronization picture. With increasing strength of the coupling, the middle clusters also appear [Fig. 7(b) and 7(c)] but their structure is somewhat different from that formed when integrating the system (1). As seen from Figs. 7(b) and 7(c), the extreme clusters are extended while the middle ones become shorter. The height of the clusters' steps, i.e., the difference between the frequencies of neighboring clusters, is less than that for system (1) and decreases rapidly as the strength of coupling increases. Thus, the region of cluster synchronization significantly shrinks when only phase dynamics is taken into consideration. Moreover, in this case the cluster structure appears to be more sensitive to noise perturbations. This is illustrated in Fig. 4(d) when a weak noise of intensity $D=0.000\,01$ is added to the system (4). As follows from the figure, the noise causes the middle clusters to be destroyed.

VI. CONCLUSIONS

In this paper we have numerically studied the dynamics of a chain of diffusively coupled Van der Pol oscillators. The numerical results obtained allow us to make a number of important conclusions.

Cluster synchronization observed in a chain of nonidentical elements appears to be sufficiently stable against uncorrelated Gaussian fluctuations added to each element. The cluster structure can be considerably destroyed in the presence of noise of large intensities.

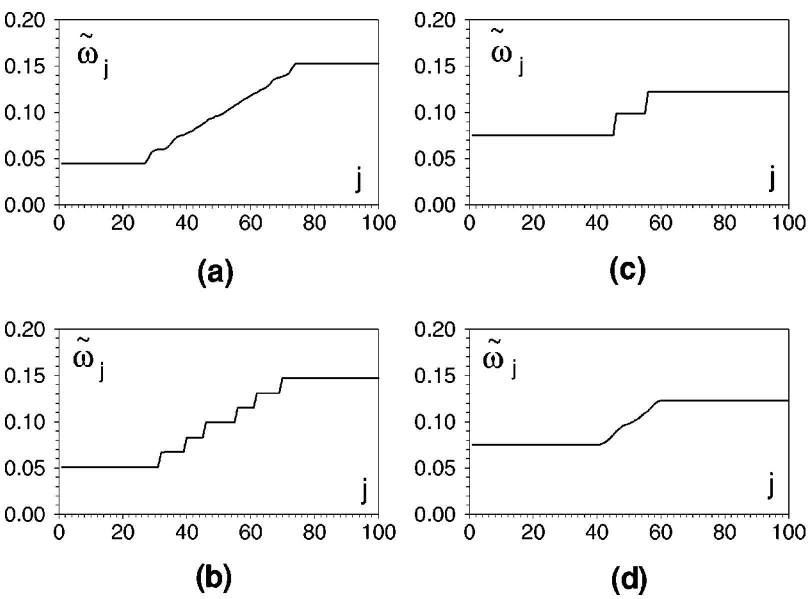

FIG. 7. Distributions of the perturbed frequencies in the chain, described by the phase equations (4), for $\Delta=0.002$ and for different strengths of the coupling: (a) $g=0.55$, (b) $g=0.7$, (c) $g=1.5$ without noise, and (d) $g=1.5$ in the presence of noise with intensity $D=0.000\,01$.

A synchronous external forcing injected even in individual elements of the chain can partially recover the initial cluster structure even in the presence of enough large noise.

The notion of effective synchronization introduced in [44] for a single Van der Pol oscillator and characterized by the effective diffusion coefficient D_{eff} can be extended to the case of synchronized clusters. With this, the effective size of a cluster at the given noise level is determined by the value of D_{eff}.

The application of an external periodic signal to individual elements of the chain causes the lengths and the characteristic frequencies of clusters to change. The most noticeable changes in the cluster structure can be distinguished under asynchronous forcing when the controlling signal frequency is larger or less than the characteristic frequency of the controlled cluster.

Cluster synchronization can be realized in a chain of identical oscillators by injecting external signals in certain elements.

Amplitude dynamics may play an essential role in creating the cluster structure. Cluster synchronization can also be observed in a chain modeled by the phase equations only. But this effect is realized in a considerably narrow range of coupling parameter values. Besides, the cluster structure appears to be more sensitive to noise perturbations.

ACKNOWLEDGMENTS

We gratefully acknowledge fruitful and valuable discussions with Dr. A. Bulsara and Dr. V. Astakhov. This work was supported by the Naval Research Laboratory under Contract No. N68171-00-M-5430.

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